Stochastic Processes and their Applications 45 (1993) 141-154
North-Holland

Interchangeability of expectation and differentiation of waiting times in GI/G/1 queues

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Received 29 May 1989
Revised 4 February 1991

A family of stable GI/G/1 queues, whose service time distributions depend on a real-valued parameter, θ, is considered. Let $z_n(\theta, \omega)$ denote a realization of the waiting time of the nth customer in the θ-dependent queue, for a sample sequence ω in the underlying probability space. Let $Z(\theta)$ denote the expected value of waiting time in the θ-dependent queue, that is, the queue with the θ-dependent service time distribution. Under appropriate conditions, the following will be shown: (1) Z is a continuously differentiable function of θ; (2) for almost every ω, $\partial z_n(\theta, \omega)/\partial\theta$ exists for every $n = 1, 2, \ldots$, and as $N \to \infty$, $\sum_{n=1}^{N} (\partial z_n(\theta, \omega)/\partial\theta)/N \to \partial Z(\theta)/\partial\theta$. These properties are important in simulation-based optimization of functions of θ, involving the average customer's waiting time in GI/G/1 queues.

GI/G/1 queue * sensitivity analysis * perturbation analysis

1. Introduction and problem statement

Consider a family of stable FIFO GI/G/1 queues, parametrized by a real-valued number, θ, where $\theta \in \Gamma$, a bounded closed interval. All of the queues have the same interarrival random process. Let ξ be the interarrival random variable, defined on a probability space, $(\bar{\Omega}_1, \bar{F}_1, \bar{p}_1)$. The service times of the queues are random variables that depend on θ, and they are denoted by $x(\theta)$. For every $\theta \in \Gamma$, $x(\theta)$ is defined on a probability space $(\bar{\Omega}_2, \bar{F}_2, \bar{p}_2)$, independent of θ. ξ can be thought of as a real-valued function defined on $\bar{\Omega}_1$, whose argument is $\bar{\omega}_1 \in \bar{\Omega}_1$. $x(\theta)$ can be thought of as a real-valued function defined on $\bar{\Omega}_2$, whose argument is $\bar{\omega}_2 \in \bar{\Omega}_2$. Actually, $x(\cdot)$ can be thought of as a function defined on $\Gamma \times \bar{\Omega}_2$, whose argument is $(\theta, \bar{\omega}_2)$. Fix a $\theta \in \Gamma$. Suppose that one wishes to simulate the θ-dependent queue (i.e., the queue whose service times are $x(\theta, \cdot)$) in order to estimate the mean steady state customer's waiting time, denoted by $Z(\theta)$. Starting from some initial load, denoted by z_1 (which can be determined in an arbitrary way — fixed to a given value, or drawn according to some distribution), one can compute the waiting times, $z_n(\theta)$, by the Lindley equation [2], namely $z_{n+1}(\theta) = \max(z_n(\theta) + x(\theta, \bar{\omega}_{2,n}) - \xi(\bar{\omega}_{1,n}), 0)$,

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$n=2,3,\ldots$, with $z_1(\theta)=z_1$; here $z_n(\theta)$ is the waiting time of the nth customer, and $\bar{\omega}_{1,n}$ and $\bar{\omega}_{2,n}$ are the nth drawn sample points from $\bar{\Omega}_1$ and $\bar{\Omega}_2$ for the nth interarrival time and service time, respectively. It can be seen that the waiting time sequence, $z_n(\theta)$, depends on θ and $\bar{\omega}_{i,n}$, $i=1,2$, $n=1,2,\ldots$. Let (Ω, F, p) denote the probability space defined as the countable Cartesian product of $(\bar{\Omega}_1, \bar{F}_1, \bar{p}_1)\times(\bar{\Omega}_2, \bar{F}_2, \bar{p}_2)$. Let a typical point $\omega\in\Omega$ have the form $\omega=(\bar{\omega}_{1,1}, \bar{\omega}_{2,1}, \bar{\omega}_{1,2}, \bar{\omega}_{2,2}, \ldots)$. Then, the sequence $z_n(\theta)$, $n=1,2,\ldots$, depends on θ, z_1, and $\omega\in\Omega$. We henceforth will assume that z_1 is fixed. We will denote $z_1(\theta)=z_1$, $x_n(\theta)=x(\theta,\bar{\omega}_{2,n})$, $\xi_n=\xi(\bar{\omega}_{1,n})$, and let C_n denote the nth customer. Then, by the Lindley equation, for every $n=1,2,\ldots$,

$$z_{n+1}(\theta)=\max(z_n(\theta)+x_n(\theta)-\xi_n, 0). \tag{1}$$

In the forthcoming, 'prime' will denote derivative with respect to θ. The following assumptions will be made:

Assumption 1.1. The random variable ξ has a bounded density function.

Assumption 1.2. For every $\theta\in\Gamma$ and $\bar{\omega}_2\in\bar{\Omega}_2$ the function $\theta\to x(\theta,\bar{\omega}_2)$ is twice continuously differentiable with respect to θ, and $x'(\theta,\bar{\omega}_2)\geqslant 0$. There exists a constant $K>0$ such that for every $\theta\in\Gamma$ and $\bar{\omega}_2\in\bar{\Omega}_2$, $x(\theta,\bar{\omega}_2)+x'(\theta,\bar{\omega}_2)+|x''(\theta,\bar{\omega}_2)|\leqslant K$.

Assumption 1.3. There exists a bounded continuously differentiable function $\phi:\mathbb{R}^+\times\Gamma\to\mathbb{R}^+$ such that for every $\theta\in\Gamma$ and $\bar{\omega}_2\in\bar{\Omega}_2$, $x'(\theta,\bar{\omega}_2)=\phi(x(\theta,\omega_2),\theta)$. Let $\theta_{\max}$ denote the right boundary point of Γ, and let $\theta_{\min}$ denote the left boundary point of Γ.

Assumption 1.4. (i) $0<E[x_n(\theta_{\min})]$ and $E[x_n(\theta_{\max})]<E[\xi_n]$ (i.e., the queue is stable), (ii) $E[x_n(\theta_{\max})^3]<\infty$, (iii) $E[\xi_n^3]<\infty$.

Let a cycle of the θ-dependent queue be defined as the time-interval between two consecutive instances when the queue becomes idle.

Corollary 1.1. *Under Assumption* 1.4, *for every* $\theta\in\Gamma$, *the third moments of the size (number of customers) and duration of cycles in the* θ*-dependent queue, in steady state, are finite.* □

This was proved in [5, Corollary 1]. Let C denote the maximum of the third moment of the size of cycles, and the third moment of the duration of cycles, of the $\theta_{\max}$-dependent queue. By Assumption 1.2, $x'(\cdot)\geqslant 0$, hence the service times are monotone-nondecreasing functions of θ. Therefore, for every $\theta\in\Gamma$, the third moments of size and duration of the θ-dependent queue are bounded from above by C.

Remark 1.1. It can be seen that Assumption 1.2 is satisfied for a large class of families of queues, where the service time random variables have truncated (θ-dependent) exponential distributions, deterministic and uniform distributions, or where θ is a scale or location parameter of a given distribution; see [3] for a more detailed discussion. These families of queues can also satisfy Assumption 1.3 (see [4]). In addition, if Assumption 1.2 is satisfied, and for every $\theta \in \Gamma$ the distribution function of $x(\theta)$ is monotone-increasing on its support, then Assumption 1.3 is satisfied (see [4]). The purpose of Assumption 1.3 is to guarantee that the process $z_n'(\theta)$ is ergodic and asymptotically stationary. Assumption 1.4 is needed for technical arguments in the proofs.

Now, let us return to the simulation. One can estimate $Z(\theta)$ by taking averages of $z_n(\theta)$, since for every $\theta \in \Gamma$, a.s., as $N \to \infty$, $\sum_{n=1}^{N} z_n(\theta)/N \to Z(\theta)$. This paper is concerned with an estimation of $Z'(\theta)$. First, the existence of the latter derivative has to be ascertained. We will prove:

Proposition 1.1. *The function $Z(\cdot)$ is continuously differentiable throughout Γ.*

Now, by Assumption 1.1, Assumption 1.2, and (1), for every $\theta \in \Gamma$, for almost every $\omega \in \Omega$, and for every $n = 1, 2, \ldots$, $z_n'(\theta)$ exists, and

$$z_{n+1}'(\theta) = \begin{cases} z_n'(\theta) + x_n'(\theta) & \text{if } z_n(\theta) + x_n(\theta) - \xi_n > 0, \\ 0 & \text{if } z_n(\theta) + x_n(\theta) - \xi_n < 0. \end{cases} \tag{2}$$

($z_1'(\theta) = 0$ since z_1 is assumed to be fixed.) If $z_n'(\theta)$ exists, then $z_{n+1}'(\theta)$ would not exist only if $z_n(\theta) + x_n(\theta) - \xi_n = 0$. To overcome this difficulty, we will define a sequence of (θ, ω)-dependent functions, g_n, $n = 1, 2, \ldots$. As previously done, we will suppress their dependence on ω, and denote them by $g_n(\theta)$. They are defined as follows: $g_1(\theta) = 0$. For every $n = 1, 2, \ldots$,

$$g_{n+1}(\theta) = \begin{cases} g_n(\theta) + x_n'(\theta) & \text{if } z_n(\theta) + x_n(\theta) - \xi_n > 0, \\ 0 & \text{if } z_n(\theta) + x_n(\theta) - \xi_n \leq 0. \end{cases} \tag{3}$$

Notice that $g_n(\theta) = z_n'(\theta)$ whenever the latter derivative exists, but $g_n(\theta)$ is always well-defined.

A question of primary interest in this paper is whether for every $\theta \in \Gamma$, a.s. (ω), as $N \to \infty$,

$$\sum_{n=1}^{N} g_n(\theta)/N \to Z'(\theta). \tag{4}$$

We will prove a stronger result:

Proposition 1.2. *There exists an event $\Omega_0 \subset \Omega$ such that $p(\Omega_0) = 0$ and for every $\omega \in \Omega_0^c$ and $\theta \in \Gamma$, (4) is satisfied.*

The statement that for every $\theta \in \Gamma$, (4) is a.s. satisfied amounts to strong consistency of infinitesimal analysis (IPA) [1, 3].

This proposition is stronger than strong consistency, since it asserts that a.s., (4) is satisfied for every $\theta \in \Gamma$. We believe that Proposition 1.2 or a similar assertion, stronger than strong consistency of IPA, could be needed for proving convergence of simulation-based algorithms for optimization of functions involving the mean waiting times in GI/G/1 queues. For instance, strong consistency alone did not suffice for the analysis of such an algorithm, in [7].

Strong consistency was proved in [8]. The main relevant differences between [8] and this paper are the following: First, Proposition 1.2 is stronger than strong consistency. Second, there is a difference in the assumptions made: the assumptions in [8] are essentially our Assumptions 1.1–1.3, plus the assumption that $Z(\cdot)$ is differentiable.[1] Verifying Assumptions 1.1–1.3 is usually easy, since they are about the interarrival time and service time distributions. By contrast, differentiability of Z is usually hard to ascertain, since it is an assumption about steady state quantities. Here, it is proved. Last, the analysis here and in [8] are based on different principles. In [8], mainly sample path arguments are used, much in the way it was done in [4]. Here, we start with steady state arguments, showing that $\lim_{N\to\infty} \sum_{n=1}^{N} g_n(\theta)/N$ is continuous in θ. Then, we use this continuity to establish that (4) is satisfied and that $Z'(\theta) = \lim_{N\to\infty} \sum_{n=1}^{N} g_n(\theta)/N$.

The analysis is carried out in Section 2. Section 3 is a conclusion. The Appendix contains some proofs.

2. Analysis

Recall that C_n denotes the nth customer. For every $\theta \in \Gamma$ and $\omega \in \Omega$, C_n belongs to some busy period. Let $k(n, \theta)$ denote the integer k such that C_k is the first customer in the busy period containing C_n. $k(n, \theta)$ also depends on ω, but this dependence is suppressed in the notation that follows. If $z_n(\theta) + x_n(\theta) - \xi_n = 0$, namely if C_n departs at the time C_{n+1} arrives, then we consider C_n and C_{n+1} to belong to two different busy periods (by Assumption 1.1, the probability of this event is 0). By (3), if $k(n, \theta) < n$, then

$$g_n(\theta) = \sum_{j=k(n,\theta)}^{n-1} x_j'(\theta), \tag{5}$$

and if $k(n, \theta) = n$, then $g_n(\theta) = 0$.

Let $\psi : \mathbb{R}^+ \times \Gamma \to \mathbb{R}^+$ be a bounded continuously differentiable function. For every $n = 1, 2, \ldots,$ define the θ- (and ω-) dependent function, $g_n^{\psi}(\theta)$, in the following way: If $k(n, \theta) < n$ then

$$g_n^{\psi}(\theta) = \sum_{j=k(n,\theta)}^{n-1} \psi(x_j(\theta), \theta), \tag{6}$$

[1] According to the version of [8] the author had at the time this paper was submitted. A later version contains a proof of differentiability of Z.

and if $k(n, \theta) = n$, then $g_n^\psi(\theta) = 0$. For a fixed $\theta \in \Gamma$, the process $g_n^\psi(\theta)$, $n = 1, 2, \ldots$, is stable and ergodic (see [6, Chapter 7]), and by Corollary 1.1, there exists a number $G^\psi(\theta) \in R^+$ such that, a.s., as $N \to \infty$,

$$\sum_{n=1}^{N} g_n^\psi(\theta)/N \to G^\psi(\theta). \tag{7}$$

We will consider $G^\psi(\cdot)$ as a function of θ.

Lemma 2.1. *The function $\theta :\to G^\psi(\theta)$ is continuous throughout Γ.*

The proof is in the appendix.

By Assumption 1.2 and Assumption 1.3, ϕ has the above properties of ψ: it is bounded and continuously differentiable. By (5), (6), and Assumption 1.3, $g_n^\phi(\theta) = g_n(\theta)$. Therefore, we will henceforth use the notation $g_n^\phi(\theta)$ for $g_n(\theta)$. By (2) and (5), applying lemma 2.1 to ϕ, we see that $G^\phi(\theta) = \lim_{N\to\infty} \sum_{n=1}^{N} z'_n(\theta)/N$ is a continuous function of θ. It still remains to show that $G^\phi(\theta) = Z'(\theta)$.

Define the function ψ by $\psi(x, \theta) = 1$ for every $(x, \theta) \in \mathbb{R}^+ \times \Gamma$. We will use (6) with ψ and with ϕ. By Lemma 2.1, $G^\phi(\cdot)$ and $G^\psi(\cdot)$ are continuous in Γ.

By Assumption 1.2, there exists a $K > 0$ such that for all $\theta \in \Gamma$ and $\bar{\omega}_2 \in \bar{\Omega}_2$, $|x''(\theta)| \leq K$. Consider an $\varepsilon > 0$, and set $\Delta_\varepsilon = \varepsilon/K$. The following technical lemma will be used to relate the sample path quantities to the steady state functions G^ϕ and G^ψ.

Lemma 2.2. *For every $\theta \in \Gamma$, $\delta \in [0, \Delta_\varepsilon]$ satisfying $\theta + \delta \in \Gamma$, $t \in [0, 1]$, $\omega \in \Omega$, and $n = 1, 2, \ldots$,*

$$\delta(g_n^\phi(\theta) - \varepsilon g_n^\psi(\theta)) \leq z_n(\theta+\delta) - z_n(\theta) \leq \delta(g_n^\phi(\theta+\delta) + \varepsilon g_n^\psi(\theta+\delta)) \tag{8}$$

and

$$g_n^\phi(\theta) - \varepsilon g_n^\psi(\theta) \leq g_n^\phi(\theta + t\delta) \leq g_n^\phi(\theta+\delta) + \varepsilon g_n^\psi(\theta+\delta). \tag{9}$$

The proof is in the appendix. In the proof of the following lemma, K is given in Assumption 1.2, and $\Delta_\varepsilon = \varepsilon/K$.

Lemma 2.3. *For every $\theta \in \Gamma$, $Z'(\theta)$ exists and $Z'(\theta) = G^\phi(\theta)$.*

Proof. Without loss of generality, fix a θ in the interior of Γ. Fix an $\varepsilon > 0$. Fix a monotone-decreasing sequence δ_i, $i = 1, 2, \ldots$, such that $\delta_1 < \Delta_\varepsilon$, $\theta + \delta_1 \in \Gamma$, and $\delta_i \to 0$. By (8), for every i and n,

$$g_n^\phi(\theta) - \varepsilon g_n^\psi(\theta) \leq \frac{z_n(\theta+\delta_i) - z_n(\theta)}{\delta_i} \leq g_n^\phi(\theta+\delta_i) + \varepsilon g_n^\psi(\theta+\delta_i). \tag{10}$$

By (7), with ψ and ϕ, and by ergodicity of the waiting time process, for every $i=1,2,\ldots$, there exists an event $\Omega_i \subset \Omega$ such that $p(\Omega_i)=0$ and for every $\omega \in \Omega_i^c$, as $N \to \infty$, (11)-(16), below, are satisfied.

$$\sum_{n=1}^{N} g_n^{\phi}(\theta)/N \to G^{\phi}(\theta), \tag{11}$$

$$\sum_{n=1}^{N} g_n^{\psi}(\theta)/N \to G^{\psi}(\theta), \tag{12}$$

$$\sum_{n=1}^{N} g_n^{\phi}(\theta+\delta_i)/N \to G^{\phi}(\theta+\delta_i), \tag{13}$$

$$\sum_{n=1}^{N} g_n^{\psi}(\theta+\delta_i)/N \to G^{\psi}(\theta+\delta_i), \tag{14}$$

$$\sum_{n=1}^{N} z_n(\theta)/N \to Z(\theta), \tag{15}$$

$$\sum_{n=1}^{N} z_n(\theta+\delta_i)/N \to Z(\theta+\delta_i). \tag{16}$$

Let $\Omega_0 = \bigcup_{i=1}^{\infty} \Omega_i$. Then $p(\Omega_0)=0$ and, for every $\omega \in \Omega_0^c$, for every $i=1,2,\ldots$, (11)-(16) are satisfied. Therefore, by (8), for every $i=1,2,\ldots$,

$$G^{\phi}(\theta) - \varepsilon G^{\psi}(\theta) \leq \frac{Z(\theta+\delta_i)-Z(\theta)}{\delta_i} \leq G^{\phi}(\theta+\delta_i) + \varepsilon G^{\psi}(\theta+\delta_i). \tag{17}$$

Since G^{ϕ} and G^{ψ} are continuous (Lemma 2.1), by taking i to ∞ in (17),

$$\begin{aligned} G^{\phi}(\theta) - \varepsilon G^{\psi}(\theta) &\leq \liminf_{i\to\infty} \frac{Z(\theta+\delta_i)-Z(\theta)}{\delta_i} \\ &\leq \limsup_{i\to\infty} \frac{Z(\theta+\delta_i)-Z(\theta)}{\delta_i} \leq G^{\phi}(\theta) + \varepsilon G^{\psi}(\theta). \end{aligned} \tag{18}$$

Since ε and the sequence δ_i were arbitrary, and since $G^{\psi}(\cdot)$ is bounded on Γ (because it is continuous — see Lemma 2.1), as $\delta \to 0$ from the right,

$$\frac{Z(\theta+\delta)-Z(\theta)}{\delta} \to G^{\phi}(\theta). \tag{19}$$

This establishes the right derivative. The left derivative can be shown in a similar way. □

Proof of Proposition 1.1. By Lemma 2.3, $Z'(\cdot) = G^{\phi}(\cdot)$. By Lemma 2.1, $G^{\phi}(\cdot)$ is continuous. □

Proof of Proposition 1.2. Fix a sequence $\theta_j, j=1,2,\ldots$, dense in Γ. By (7), with ϕ and ψ, for every $j=1,2,\ldots$, there exists an event $\Omega_j \subset \Omega$ such that $p(\Omega_j)=0$ and, for every $\omega \in \Omega_j^c$, as $N \to \infty$,

$$\sum_{n=1}^{N} g_n^{\phi}(\theta_j)/N \to G^{\phi}(\theta_j) \tag{20}$$

and

$$\sum_{n=1}^{N} g_n^{\psi}(\theta_j)/N \to G^{\psi}(\theta_j). \tag{21}$$

Let $\Omega_0 = \bigcup_{j=1}^{\infty} \Omega_j$. Then $p(\Omega_0) = 0$ and for every $\omega \in \Omega_0^c$, for every $j = 1, 2, \ldots,$ (20) and (21) are satisfied. Fix an $\varepsilon > 0$. For every $\theta \in \Gamma$, let $\delta_\varepsilon = \varepsilon / K$, where K is given in Assumption 1.2. Let $j(1)$ and $j(2)$ be a pair of integers such that

$$\theta_{j(1)} \leq \theta \leq \theta_{j(2)} \leq \theta_{j(1)} + \Delta_\varepsilon.$$

Then, by (9), for every $n = 1, 2, \ldots,$

$$g_n^{\phi}(\theta_{j(1)}) - \varepsilon g_n^{\psi}(\theta_{j(1)}) \leq g_n^{\phi}(\theta) \leq g_n^{\phi}(\theta_{j(2)}) + \varepsilon g_n^{\psi}(\theta_{j(2)}). \tag{22}$$

Therefore, by (20) and (21), and by (7) with ϕ and ψ,

$$G^{\phi}(\theta_{j(1)}) - \varepsilon G^{\psi}(\theta_{j(1)}) \leq G^{\phi}(\theta) \leq G^{\phi}(\theta_{j(2)}) + \varepsilon G^{\psi}(\theta_{j(2)}). \tag{23}$$

It remains to extend (20) for every $\theta \in \Gamma$, namely, to show that for every $\theta \in \Gamma$ and $\omega \in \Omega_0^c$, as $N \to \infty$,

$$\sum_{n=1}^{N} g_n^{\phi}(\theta)/N \to G^{\phi}(\theta). \tag{24}$$

By Lemma 2.3, $G^{\phi}(\theta) = Z'(\theta)$, hence, (24) will complete the proof.

Fix an $\omega \in \Omega_0^c$ and a $\theta \in \Gamma$. By Lemma 2.1 $G^{\phi}(\cdot)$ is continuous, hence there exist integers $j(1)$ and $j(2)$ such that

$$\theta_{j(1)} \leq \theta \leq \theta_{j(2)} \leq \theta_{j(1)} + \Delta_\varepsilon$$

and

$$|G^{\phi}(\theta_{j(2)}) - G^{\phi}(\theta_{j(1)})| < \varepsilon. \tag{25}$$

Let L be an upper-bound on $G^{\psi}(\theta)$ over $\theta \in \Gamma$ (which is guaranteed by Lemma 2.1). By (20)-(22),

$$\begin{aligned} G^{\phi}(\theta_{j(1)}) - \varepsilon G^{\psi}(\theta_{j(1)}) &\leq \liminf_{N\to\infty} \sum_{n=1}^{N} g_n^{\phi}(\theta)/N \\ &\leq \limsup_{N\to\infty} \sum_{n=1}^{N} g_n^{\phi}(\theta)/N \leq G^{\phi}(\theta_{j(2)}) + \varepsilon G^{\psi}(\theta_{j(2)}). \end{aligned} \tag{26}$$

Hence, and by (25),

$$\limsup_{N\to\infty} \sum_{n=1}^{N} g_n^{\phi}(\theta)/N - \liminf_{N\to\infty} \sum_{n=1}^{N} g_n^{\phi}(\theta)/N \leq \varepsilon(1+2L). \tag{27}$$

Since ε can be arbitrarily small, the limit $\lim_{N\to\infty}\sum_{n=1}^{N} g_n^{\phi}(\theta)/N$ exists. By (26), the fact that $\theta_{j(1)}$ and $\theta_{j(2)}$ can be arbitrary close to θ, and continuity of G^{ϕ}, the above limit is equal to $G^{\phi}(\theta)$. This establishes (24), and completes the proof. □

3. Conclusions

This paper contains an analysis of FIFO GI/G/1 queues, whose service time distributions depend on a real-valued parameter, θ. It is shown that the steady state average customer's waiting time is a continuously differentiable function of θ, and that its derivative can be estimated by sample path averages. This suggests that such estimates can be used in simulation-based optimization.

4. Appendix

We now prove Lemma 2.1 and Lemma 2.2.

Proof of Lemma 2.1. For the sake of simplicity, but without loss of generality, we will assume that the queue starts empty. By Assumption 1.2, the size of the first busy period of the queue, as a function of θ, is samplewise monotone-nondecreasing in θ. We will show the following: For every $\varepsilon > 0$ there exists a $\bar{\delta} > 0$ such that for every $\delta \in (0, \bar{\delta})$ and for every $\theta \in \Gamma$,

$$G^{\psi}(\theta) - G^{\psi}(\theta - \delta) < \varepsilon \tag{28}$$

(implicitly assuming that $\theta - \delta \in \Gamma$).

First, we establish some notation. Let C_n denote the nth customer, $Q(\theta)$—the queue with θ fixed, $d_n(\theta)$—the departure time of C_n from $Q(\theta)$, a_n—the arrival time of C_n to the queue (it is independent of θ), $B(\theta)$—the first busy period of $Q(\theta)$ (i.e., the customers it contains), $\Omega_m(\theta)$—the event that $B(\theta)$ consists of m customers ($m = 1, 2, \ldots$), L_{ψ}—the global Lipschitz constant of $\psi(x(\cdot), \cdot)$ (which exists by Assumption 1.2, since $\mathrm{d}\psi(x(\theta), \theta)/\mathrm{d}\theta = (\partial\psi(x(\theta), \theta)/\partial x)x'(\theta) + \partial\psi(x(\theta), \theta)/\partial\theta$, which is bounded on $(\theta, \bar{\omega}_2) \in \Gamma \times \bar{\Omega}_2$), M_{ψ}—an upper bound on $\psi(x(\theta), \theta)$ over $(\theta, \bar{\omega}_2) \in \Gamma \times \bar{\Omega}_2$, K—the upper bound on $x_n(\cdot) + x_n'(\cdot) + |x_n''(\cdot)|$, given in Assumption 1.2, L_{ξ}—an upper bound on the density function of the interarrival times (see Assumption 1.1), and $S(\theta)$—the size (number of customers) of the first busy period at $Q(\theta)$. Recall that $\theta_{\max}$ is the right point of Γ. By Corollary 1.1, and since $x'(\theta) \geq 0$ (Assumption 1.2), for every $\theta \in \Gamma$, and $k = 1, 2, 3$, $E[S(\theta)^k] \leq E[S(\theta_{\max})^k]$.

By (6) and (7),

$$G^{\psi}(\theta) = \frac{E[\sum_{j=1}^{S(\theta)} \sum_{i=1}^{j-1} \psi(x_i(\theta), \theta)]}{E[S(\theta)]}. \tag{29}$$

Denote the numerator of (29) by $\Delta(\theta)$. We will show uniform continuity of G^{ψ}. To this end, we will prove that $\Delta(\cdot)$ and $E[S(\cdot)]$ are uniformly continuous, then, since $E[S(\theta)]$ is bounded from below by $E[x(\theta)]>0$ (see Assumption 1.4(i)), uniform continuity of G^{ψ} will follow.

Consider first $E[S(\theta)]$. Note that $E[S(\theta)]=\sum_{m=1}^{\infty} mp(\Omega_m(\theta))$. Fix $\theta\in\Gamma$ and $\delta>0$. Now,

$$E[S(\theta)]-E[S(\theta-\delta)]=\sum_{m=1}^{\infty} m[p(\Omega_m(\theta))-p(\Omega_m(\theta-\delta))]$$

$$\leq \sum_{m=1}^{\infty} mp(\Omega_m(\theta)\cap\Omega_m(\theta-\delta)^c). \tag{30}$$

We now show that

$$p(\Omega_m(\theta)\cap\Omega_m(\theta-\delta)^c)\leq L_\xi m^2K\delta. \tag{31}$$

Fix $\omega\in\Omega_m(\theta)\cap\Omega_m(\theta-\delta)^c$. Then, $S(\theta)=m$ and $S(\theta-\delta)<m$. This means that $C_m\in B(\theta)$, but $C_m\in B(\theta-\delta)^c$ (recall that $B(\theta)$ is the first busy period in $Q(\theta)$). Let $j=S(\theta-\delta)$, i.e., C_j is the last customer in the first busy period of $Q(\theta-\delta)$. Then, $j<m$. Now, C_{j+1} arrives to $Q(\theta-\delta)$ after C_j departs. Therefore, $a_{j+1}\geq d_j(\theta-\delta)$. But $j+1\leq m$, hence $C_{j+1}\in B(\theta)$. Therefore, C_{j+1} arrives to $Q(\theta)$ before C_j departs. This means that $a_{j+1}<d_j(\theta)$. Hence,

$$d_j(\theta-\delta)\leq a_{j+1}<d_j(\theta). \tag{32}$$

Now, for every $i=1,\dots,j$, $C_i\in B(\theta)\cap B(\theta-\delta)$. Hence, and since $x_i'\geq 0$, for every $\eta\in[\theta-\delta,\theta]$, $C_i\in B(\eta)$. This implies that for every $i=1,\dots,j$, $d_i(\eta)=d_{i-1}(\eta)+x_i(\eta)$, and, since $d_1(\eta)=a_1+x_1(\eta)$, we have that $d_j(\eta)=\sum_{i=1}^{j} x_i(\eta)$. Therefore,

$$d_j'(\eta)=\sum_{i=1}^{j} x_i'(\eta)\leq jK\leq mK. \tag{33}$$

Now, $d_j(\cdot)$ is a.s. continuous and piecewise differentiable. Therefore, (32), (33), and the mean value theorem imply that

$$d_j(\theta)-mK\delta\leq a_{j+1}<d_j(\theta). \tag{34}$$

Let $\bar{\Omega}_{m,j}(\theta)$ denote the event that (34) is satisfied. We have just seen that if $\omega\in\Omega_m(\theta)\cap\Omega_m(\theta-\delta)^c$, then (34) is satisfied for some $j=1,\dots,m-1$, i.e., $\omega\in\bigcup_{j=1}^{m-1}\bar{\Omega}_{m,j}(\theta)$. Hence,

$$\Omega_m(\theta)\cap\Omega_m(\theta-\delta)^c\subset\bigcup_{j=1}^{m-1}\bar{\Omega}_{m,j}(\theta). \tag{35}$$

By (34) and Assumption 1.1, $p(\bar{\Omega}_{m,j}(\theta))\leq L_\xi mK\delta$, therefore, and by (35), (31) is satisfied.

Fix $\varepsilon>0$. By Corollary 1.1, $E[S(\theta_{\max})^2]<\infty$, and since $x'(\cdot)\geq 0$, $E[S(\theta)^2]\leq E[S(\theta_{\max})^2]$. Choose an integer M such that $E[S(\theta_{\max})^2]/M<\frac{1}{2}\varepsilon$. Then,

$$\sum_{m=M+1}^{\infty} mp(\Omega_m(\theta))\leq\frac{1}{M}\sum_{m=M+1}^{\infty} m^2p(\Omega_m(\theta))\leq\frac{1}{M}\sum_{m=1}^{\infty} m^2p(\Omega_m(\theta))$$
$$=\frac{1}{M}E[S(\theta)^2]\leq\frac{1}{M}E[S(\theta_{\max})^2]<\tfrac{1}{2}\varepsilon.$$

Therefore,

$$\sum_{m=M+1}^{\infty} mp(\Omega_m(\theta)\cap\Omega_m(\theta-\delta)^{\mathrm{c}})<\tfrac{1}{2}\varepsilon. \tag{36}$$

Choose a $\delta>0$ such that $\sum_{m=1}^{M} L_\xi m^3K\delta<\frac{1}{2}\varepsilon$. Then, by (31),

$$\sum_{m=1}^{M} mp(\Omega_m(\theta)\cap\Omega_m(\theta-\delta)^{\mathrm{c}})<\tfrac{1}{2}\varepsilon. \tag{37}$$

By (30), (36) and (37),

$$E[S(\theta)]-E[S(\theta-\delta)]\leq\varepsilon. \tag{38}$$

Note that δ depends on M, which depends on ε, but neither ε nor M depend on $\theta\in\Gamma$. Since $x'(\cdot)\geq 0$, $E[S(\theta)]-E[S(\theta-\delta)]\geq 0$. This, with (38), establish that $E[S(\cdot)]$ is uniformly continuous.

Next, we turn to

$$\Delta(\theta)=E\left[\sum_{j=1}^{S(\theta)}\sum_{i=1}^{j-1}\psi(x_i(\theta),\theta)\right].$$

Fix $\theta\in\Gamma$ and $\delta>0$. Then,

$$\sum_{j=1}^{S(\theta)}\sum_{i=1}^{j-1}\psi(x_i(\theta),\theta)-\sum_{j=1}^{S(\theta-\delta)}\sum_{i=1}^{j-1}\psi(x_i(\theta-\delta),\theta-\delta)$$
$$=\sum_{j=1}^{S(\theta-\delta)}\sum_{i=1}^{j-1}(\psi(x_i(\theta),\theta)-\psi(x_i(\theta-\delta),\theta-\delta))$$
$$+\sum_{j=S(\theta-\delta)+1}^{S(\theta)}\sum_{i=1}^{j-1}\psi(x_i(\theta),\theta). \tag{39}$$

Now

$$\sum_{j=1}^{S(\theta-\delta)}\sum_{i=1}^{j-1}(\psi(x_i(\theta),\theta)-\psi(x_i(\theta-\delta),\theta-\delta))$$
$$\leq\sum_{j=1}^{S(\theta-\delta)}\sum_{i=1}^{j-1}L_\Psi\delta=L_\Psi\delta\sum_{j=1}^{S(\theta-\delta)}(j-1)\leq\tfrac{1}{2}L_\Psi\delta S(\theta-\delta)^2. \tag{40}$$

Therefore, and by the monotonicity of $x_i(\cdot)$,

$$E\left[\sum_{j=1}^{S(\theta-\delta)}\sum_{i=1}^{j-1}(\psi(x_i(\theta),\theta)-\psi(x_i(\theta-\delta),\theta-\delta))\right]\leq\tfrac{1}{2}L_\Psi\delta E[S(\theta_{\max})^2]. \tag{41}$$

Next,

$$\sum_{j=S(\theta-\delta)+1}^{S(\theta)} \sum_{i=1}^{j-1} \psi(x_i(\theta), \theta)$$

$$\leqslant \sum_{j=S(\theta-\delta)+1}^{S(\theta)} \sum_{i=1}^{j-1} M_\Psi = M_\Psi \sum_{j=S(\theta-\delta)+1}^{S(\theta)} (j-1), \tag{42}$$

and therefore,

$$E\left[\sum_{j=S(\theta-\delta)+1}^{S(\theta)} \sum_{i=1}^{j-1} \psi(x_i(\theta), \theta)\right] \leqslant M_\Psi \sum_{m=1}^{\infty} mp(\Omega_m(\theta) \cap \Omega_m(\theta-\delta)^c). \tag{43}$$

Fix $\varepsilon < 0$. Choose an integer M such that $(1/M)M_\Psi E[S(\theta_{\max})^2] < \frac{1}{2}\varepsilon$. Then, by (31),

$$M_\Psi \sum_{m=M+1}^{\infty} mp(\Omega_m(\theta) \cap \Omega_m(\theta-\delta)^c)$$

$$\leqslant \frac{1}{M} M_\Psi \sum_{m=M+1}^{\infty} m^2 p(\Omega_m(\theta_{\max})) \leqslant \tfrac{1}{2}\varepsilon.$$

Therefore, and by (43) and (31),

$$\begin{aligned} E\left[\sum_{j=S(\theta-\delta)+1}^{S(\theta)} \sum_{i=1}^{j-1} \psi(x_i(\theta), \theta)\right] &\leqslant M_\Psi \sum_{m=1}^{M} m^3 L_\xi K\delta + M_\Psi \sum_{m=M+1}^{\infty} mp(\Omega_m(\theta)) \\ &\leqslant M_\Psi \sum_{m=1}^{M} m^3 L_\xi K\delta + \frac{M_\Psi}{M} E[S(\theta)^2] \\ &< M_\Psi \sum_{m=1}^{M} m^3 L_\xi K\delta + \tfrac{1}{2}\varepsilon. \end{aligned} \tag{44}$$

Now, by (39), (41), and (44),

$$\Delta(\theta) - \Delta(\theta-\delta) \leqslant \tfrac{1}{2} L_\Psi \delta E[S(\theta_{\max})^2] + M_\Psi \sum_{m=1}^{M} m^3 L_\xi K\delta + \tfrac{1}{2}\varepsilon. \tag{45}$$

Next, choose $\delta > 0$ such that

$$\tfrac{1}{2} L_\Psi \delta E[S(\theta_{\max})^2] + M_\Psi \sum_{m=1}^{M} m^3 L_\xi K\delta < \tfrac{1}{2}\varepsilon.$$

Then, and by (45),

$$\Delta(\theta) - \Delta(\theta-\delta) < \varepsilon. \tag{46}$$

An inequality concerning $\delta < 0$ can be derived in a similar way. This establishes the uniform continuity of Δ, and completes the proof of the lemma. □

Proof of Lemma 2.2. Recall that $|x''(\theta)| \leqslant K$ for every $\theta \in \Gamma$ and $\bar{\omega}_2 \in \bar{\Omega}_2$, and that $\Delta_\varepsilon = \varepsilon / K$. Recall that $\psi(x, \theta) = 1$ for every $(x, \theta) \in \mathbb{R}^+ \times \Gamma$.

Fix $\theta \in \Gamma$, $t \in [0, 1]$, and $\delta \in [0, \Delta_\varepsilon]$ such that $\theta + \delta \in \Gamma$. By the mean value theorem, $|x'(\theta + t\delta) - x'(\theta)| \leqslant \delta K \leqslant \Delta_\varepsilon K = \varepsilon$, and $|x'(\theta + \delta) - x'(\theta + t\delta)| \leqslant \delta K \leqslant \Delta_\varepsilon K = \varepsilon$. Hence,

$$x'(\theta + t\delta) + \varepsilon \geqslant x'(\theta) \tag{47}$$

and

$$x'(\theta + \delta) + \varepsilon \geqslant x'(\theta + t\delta). \tag{48}$$

Recall that $k(n, \theta)$ is the integer k such that C_k is the first customer in the busy period of $Q(\theta)$, containing C_n, $g_n^\phi(\theta) = \sum_{i=k(n,\theta)}^{n-1} x_i'(\theta)$, and, since $\psi(x, \theta) = 1$, and by (6), $g^\psi(\theta) = \sum_{i=k(n,\theta)}^{n-1} \psi(x_i(\theta), \theta) = n - k(n, \theta)$. By Assumption 1.2, $x'(\theta) \geqslant 0$. Hence, if $\theta(1) < \theta(2)$, then for every $k = 1, 2, \ldots, x_k(\theta(1)) \leqslant x_k(\theta(2))$. Therefore, busy periods of $Q(\theta(1))$ are contained in busy periods of $Q(\theta(2))$, hence, for every $n = 1, 2, \ldots, k(n, \theta(1)) \geqslant k(n, \theta(2))$. Now, (9) can be seen as follows:

$$\begin{aligned} g_n^\phi(\theta + t\delta) &= \sum_{i=k(n,\theta+t\delta)}^{n-1} x_i'(\theta + t\delta) \\ &\geqslant \sum_{i=k(n,\theta)}^{n-1} x_i'(\theta + t\delta) \quad (\text{since } k(n, \theta) \geqslant k(n, \theta + t\delta)) \\ &\geqslant \sum_{i=k(n,\theta)}^{n-1} (x_i'(\theta) - \varepsilon) \quad (\text{by (47)}) \\ &= g_n^\phi(\theta) - \varepsilon g_n^\psi(\theta), \end{aligned} \tag{49}$$

which shows the left inequality of (9). The right inequality can be derived in a similar way:

$$\begin{aligned} g_n^\phi(\theta + t\delta) &= \sum_{i=k(n,\theta+t\delta)}^{n-1} x_i'(\theta + t\delta) \leqslant \sum_{i=k(n,\theta+\delta)}^{n-1} x_i'(\theta + t\delta) \\ &\leqslant \sum_{i=k(n,\theta+\delta)}^{n-1} (x_i(\theta + \delta) + \varepsilon) = g_n^\phi(\theta + \delta) + \varepsilon g_n^\psi(\theta + \delta). \end{aligned} \tag{50}$$

This establishes (9).

(8) is proved by induction. For $n = 1$, all of the terms in (8) are equal to 0. Suppose that (8) is satisfied for n. We will show it for $n + 1$. There are four possible cases:

(a) $z_n(\theta) + x_n(\theta) - \xi_n > 0$ and $z_n(\theta + \delta) + x_n(\theta + \delta) - \xi_n > 0$,

(b) $z_n(\theta) + x_n(\theta) - \xi_n > 0$ and $z_n(\theta + \delta) + x_n(\theta + \delta) - \xi_n \leqslant 0$,

(c) $z_n(\theta) + x_n(\theta) - \xi_n \leqslant 0$ and $z_n(\theta + \delta) + x_n(\theta + \delta) - \xi_n > 0$,

(d) $z_n(\theta) + x_n(\theta) - \xi_n \leqslant 0$ and $z_n(\theta + \delta) + x_n(\theta + \delta) - \xi_n \leqslant 0$.

Consider first (a). C_{n+1} belongs to the same busy period as C_n in both $Q(\theta)$ and $Q(\theta+\delta)$. Hence, $k(n+1,\theta)=k(n,\theta)$ and $k(n+1,\theta+\delta)=k(n,\theta+\delta)$. Therefore, by (5) and (6), $g^{\phi}_{n+1}(\theta)=g^{\phi}_n(\theta)+x'_n(\theta)$, $g^{\phi}_{n+1}(\theta+\delta)=g^{\phi}_n(\theta+\delta)+x'_n(\theta+\delta)$, $g^{\psi}_{n+1}(\theta)=g^{\psi}_n(\theta)+1$, and $g^{\psi}_{n+1}(\theta+\delta)=g^{\psi}_n(\theta+\delta)+1$. Also, by (1), $z_{n+1}(\theta)=z_n(\theta)+x_n(\theta)-\xi_n$, and $z_{n+1}(\theta+\delta)=z_n(\theta+\delta)+x_n(\theta+\delta)-\xi_n$.

By the mean value theorem there exists a $t\in[0,1]$ such that $x_n(\theta+\delta)-x_n(\theta)=\delta x'_n(\theta+t\delta)$, hence,

$$\begin{aligned}
&z_{n+1}(\theta+\delta)-z_{n+1}(\theta)\\
&\quad=z_n(\theta+\delta)+x_n(\theta+\delta)-\xi_n-(z_n(\theta)+x_n(\theta)-\xi_n)\\
&\quad=z_n(\theta+\delta)-z_n(\theta)+x_n(\theta+\delta)-x_n(\theta)\\
&\quad\leq\delta[g^{\phi}_n(\theta+\delta)+\varepsilon g^{\psi}_n(\theta+\delta)]+x_n(\theta+\delta)-x_n(\theta)\quad\text{(by (8), for } n)\\
&\quad=\delta[g^{\phi}_n(\theta+\delta)+\varepsilon g^{\psi}_n(\theta+\delta)+x'_n(\theta+t\delta)]\\
&\quad\leq\delta[g^{\phi}_n(\theta+\delta)+\varepsilon g^{\psi}_n(\theta+\delta)+x'_n(\theta+\delta)+\varepsilon]\quad\text{(by (48))}\\
&\quad=\delta[g^{\phi}_{n+1}(\theta+\delta)+\varepsilon g^{\psi}_{n+1}(\theta+\delta)].
\end{aligned}\tag{51}$$

This shows the right inequality of (8) for $n+1$. The left inequality follows in a similar way.

Consider (b). This is an impossible situation, since (by Assumption 1.2) for every $k=1,2,\ldots,x_k(\theta+\delta)\geq x_k(\theta)$, implying that $z_n(\theta+\delta)+x_n(\theta+\delta)\geq z_n(\theta)+x_n(\theta)$.

Consider (c). It implies that $k(n+1,\theta)=n+1$, hence $g^{\phi}_{n+1}(\theta)=g^{\psi}_{n+1}(\theta)=0$. Therefore, by the monotonicity of x_{n+1}, the left inequality of (8) is satisfied with $n+1$. Next, $k(n+1,\theta+\delta)=k(n,\theta+\delta)$, hence, $g^{\phi}_{n+1}(\theta+\delta)=g^{\phi}_n(\theta+\delta)+x'_n(\theta+\delta)$, $g^{\psi}_{n+1}(\theta+\delta)=g^{\psi}_n(\theta+\delta)+1$, $z_{n+1}(\theta+\delta)=z_n(\theta+\delta)+x_n(\theta+\delta)-\xi_n$, and $z_{n+1}(\theta)=0$. Therefore, by the mean value theorem, there exists a $t\in[0,1]$ such that $x_n(\theta+\delta)-x_n(\theta)=\delta x'_n(\theta+t\delta)$, hence,

$$\begin{aligned}
&z_{n+1}(\theta+\delta)-z_{n+1}(\theta)\\
&\quad=z_n(\theta+\delta)+x_n(\theta+\delta)-\xi_n\\
&\quad=z_n(\theta+\delta)-z_n(\theta)+x_n(\theta+\delta)-x_n(\theta)+z_n(\theta)+x_n(\theta)-\xi_n.
\end{aligned}\tag{52}$$

By the condition in (c), $z_n(\theta)+x_n(\theta)-\xi_n\leq 0$, hence, and by (52),

$$\begin{aligned}
&z_{n+1}(\theta+\delta)-z_{n+1}(\theta)\\
&\quad\leq z_n(\theta+\delta)-z_n(\theta)+x_n(\theta+\delta)-x_n(\theta)\\
&\quad\leq\delta[g^{\phi}_n(\theta+\delta)+\varepsilon g^{\psi}_n(\theta+\delta)]+x_n(\theta+\delta)-x_n(\theta)\quad\text{(by (8) for } n)\\
&\quad=\delta[g^{\phi}_n(\theta+\delta)+\varepsilon g^{\psi}_n(\theta+\delta)+x'_n(\theta+t\delta)]\\
&\quad\leq\delta[g^{\phi}_n(\theta+\delta)+\varepsilon g^{\psi}_n(\theta+\delta)+x'_n(\theta+\delta)+\varepsilon]\quad\text{(by (48))}\\
&\quad=\delta[g^{\phi}_{n+1}(\theta+\delta)+\varepsilon g^{\psi}_{n+1}(\theta+\delta)].
\end{aligned}\tag{53}$$

This establishes (8) for $n+1$.

Finally, consider (d). $z_{n+1}(\theta) = z_{n+1}(\theta+\delta) = 0$, and $k(n+1, \theta+\delta) = k(n+1, \theta) = n+1$. Hence, $g^{\phi}_{n+1}(\theta) = g^{\psi}_{n+1}(\theta) = g^{\phi}_{n+1}(\theta+\delta) = g^{\psi}_{n+1}(\theta+\delta) = 0$. This shows (8), for $n+1$, and completes the proof of the lemma. □

References

[1] Y.C. Ho, Performance evaluation and perturbation analysis of discrete event dynamic systems, IEEE Trans. on Automat. Control AC-32(7) (1987) 563-572.

[2] L. Kleinrock, Queueing Systems, Vol. 1, Theory (Wiley, New York, 1975).

[3] R. Suri, Perturbation analysis: The state of the art and research issues explained via the GI/G/1 queue, Proc. of the IEEE 77(1) (1989) 114-137.

[4] R. Suri and M.A. Zazanis, Perturbation analysis gives strongly consistent sensitivity estimates for M/G/1 queues, Management Sci. 34(1) (1988).

[5] H. Thorisson, The queue GI/G/1: Finite moments of the cycle variables and uniform rates of convergence, Stochastic Process Appl. 19 (1985) 85-99.

[6] J. Walrand, An Introduction to Queueing networks (Prentice-Hall, Englewood Cliffs, NY, 1988).

[7] Y. Wardi, Analysis of a stochastic approximation algorithm for queueing networks optimization, manuscript, Georgia Inst. of Technol., School of Electr. Engrg. (Atlanta, GA, 1990).

[8] M.A. Zazanis and R. Suri, Estimating first and second derivatives of response time for GI/G/1 queues from a single sample path, submitted for publication.